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A separability criterion for density operators

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Abstract. We give a necessary and sufficient condition for a mixed quantum mechanical state to be separable. The criterion is formulated as a boundedness condition in terms of the greatest cross norm on the tensor product of trace class operators.

1. Introduction

The question of separability of density operators on finite-dimensional Hilbert spaces has recently been studied extensively in the context of quantum information theory, see, e.g., [1–6] and references therein. In this work we provide a simple mathematical characterization of separable density operators.

Throughout this paper the set of trace class operators on some Hilbert space $\mathcal{H}$ is denoted by $\mathcal{T}(\mathcal{H})$ and the set of bounded operators on $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$. A density operator is a positive trace class operator with trace one.

Definition 1. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces of arbitrary dimension. A density operator ϱ on the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ is called* separable *if there exist a family $\{\omega_i\}$ of positive real numbers, a family $\{\rho_i^{(1)}\}$ of density operators on $\mathcal{H}_1$ and a family $\{\rho_i^{(2)}\}$ of density operators on $\mathcal{H}_2$ such that*

$$\varrho = \sum_i \omega_i \rho_i^{(1)} \otimes \rho_i^{(2)} \tag{1}$$

where the sum converges in trace class norm.

Consider the spaces $\mathcal{T}(\mathcal{H}_1)$ and $\mathcal{T}(\mathcal{H}_2)$ of trace class operators on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Both spaces are Banach spaces when equipped with the trace class norm $\|\cdot\|_1^{(1)}$ or $\|\cdot\|_1^{(2)}$ respectively, see, e.g., Schatten [7]. In the following we shall drop the superscript and write $\|\cdot\|_1$ for both norms, slightly abusing the notation; it will always be clear from the context which norm is meant. The algebraic tensor product $\mathcal{T}(\mathcal{H}_1) \otimes_{\mathrm{alg}} \mathcal{T}(\mathcal{H}_2)$ of $\mathcal{T}(\mathcal{H}_1)$ and $\mathcal{T}(\mathcal{H}_2)$ is defined as the set of all finite linear combinations of elementary tensors $u \otimes v$, i.e. the set of all finite sums $\sum_{i=1}^n u_i \otimes v_i$ where $u_i \in \mathcal{T}(\mathcal{H}_1)$ and $v_i \in \mathcal{T}(\mathcal{H}_2)$ for all i.

It is well known that we can define a cross norm on $\mathcal{T}(\mathcal{H}_1) \otimes_{\mathrm{alg}} \mathcal{T}(\mathcal{H}_2)$ by [8]

$$\|t\|_\gamma := \inf \left\{ \sum_{i=1}^n \|u_i\|_1 \|v_i\|_1 \middle| t = \sum_{i=1}^n u_i \otimes v_i \right\} \tag{2}$$

where $t \in \mathcal{T}(\mathcal{H}_1) \otimes_{\mathrm{alg}} \mathcal{T}(\mathcal{H}_2)$ and where the infimum runs over all finite decompositions of t into elementary tensors. It is well known that $\|\cdot\|_\gamma$ majorizes any subcross seminorm on

$\mathcal{T}(\mathcal{H}_1)\otimes_{\rm alg}\mathcal{T}(\mathcal{H}_2)$ and that the completion $\mathcal{T}(\mathcal{H}_1)\otimes_\gamma\mathcal{T}(\mathcal{H}_2)$ of $\mathcal{T}(\mathcal{H}_1)\otimes_{\rm alg}\mathcal{T}(\mathcal{H}_2)$ with respect to $\|\cdot\|_\gamma$ is a Banach algebra [8].

In the following we specialize to the situation where both $\mathcal{H}_1$ and $\mathcal{H}_2$ are finite dimensional, hence $\mathcal{T}(\mathcal{H}_1) = \mathcal{B}(\mathcal{H}_1)$ and $\mathcal{T}(\mathcal{H}_2) = \mathcal{B}(\mathcal{H}_2)$. It is well known that the completion of $\mathcal{B}(\mathcal{H}_1)\otimes_{\rm alg}\mathcal{B}(\mathcal{H}_2)$ with respect to the spatial norm on $\mathcal{B}(\mathcal{H}_1)\otimes_{\rm alg}\mathcal{B}(\mathcal{H}_2)$ is equal to $\mathcal{B}(\mathcal{H}_1\otimes\mathcal{H}_2)$, see, e.g., [9, example (11.1.6)]. (The spatial norm on $\mathcal{B}(\mathcal{H}_1)\otimes_{\rm alg}\mathcal{B}(\mathcal{H}_2)$ is the norm induced by the operator norm on $\mathcal{B}(\mathcal{H}_1\otimes\mathcal{H}_2)$.) As $\mathcal{H}_1$ and $\mathcal{H}_2$ are finite dimensional, both $\mathcal{T}(\mathcal{H}_1) = \mathcal{B}(\mathcal{H}_1)$ and $\mathcal{T}(\mathcal{H}_2) = \mathcal{B}(\mathcal{H}_2)$ are nuclear. By nuclearity it follows that the completion of $\mathcal{B}(\mathcal{H}_1)\otimes_{\rm alg}\mathcal{B}(\mathcal{H}_2)$ with respect to $\|\cdot\|_\gamma$, denoted by $\mathcal{B}(\mathcal{H}_1)\otimes_\gamma\mathcal{B}(\mathcal{H}_2)$, equals $\mathcal{B}(\mathcal{H}_1\otimes\mathcal{H}_2)$. Moreover in finite dimensions all Banach space norms on $\mathcal{B}(\mathcal{H}_1\otimes\mathcal{H}_2)$, in particular the operator norm $\|\cdot\|$, the trace class norm $\|\cdot\|_1$, and the norm $\|\cdot\|_\gamma$, are equivalent, i.e. generate the same topology on $\mathcal{B}(\mathcal{H}_1\otimes\mathcal{H}_2)$.

2. The separability criterion

Lemma 2. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, and ϱ be a density operator on $\mathcal{H}_1\otimes\mathcal{H}_2$. Then $\|\varrho\|_\gamma \leqslant 1$ if and only if $\|\varrho\|_\gamma = 1$.*

Proof. This follows from $1 = \|\varrho\|_1 \leqslant \|\varrho\|_\gamma$. □

Proposition 3. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be finite-dimensional Hilbert spaces and let ϱ be a separable density operator on $\mathcal{H}_1\otimes\mathcal{H}_2$, then $\|\varrho\|_\gamma \leqslant 1$.*

Proof. Let ϱ be a separable density operator on $\mathcal{H}_1\otimes\mathcal{H}_2$, then there exist a family $\{\omega_i\}$ of positive real numbers, a family $\{\rho_i^{(1)}\}$ of density operators on $\mathcal{H}_1$ and a family $\{\rho_i^{(2)}\}$ of density operators on $\mathcal{H}_2$ such that

$$\varrho = \sum_i \omega_i \rho_i^{(1)} \otimes \rho_i^{(2)}$$

where the sum converges in trace class norm. If this sum is finite, then obviously $\|\varrho\|_\gamma \leqslant 1$. If the sum is infinite, consider the sequence $\{\varrho_n\}$ of trace class operators where $\varrho_n \equiv \sum_{i=1}^n \omega_i \rho_i^{(1)} \otimes \rho_i^{(2)}$. The sequence $\{\varrho_n\}$ converges to ϱ in trace class norm and is a Cauchy sequence with respect to $\|\cdot\|_\gamma$. Thus $\{\varrho_n\}$ converges to ϱ with respect to the norm $\|\cdot\|_\gamma$ and we have $\|\varrho_n\|_\gamma \leqslant 1$ for all n. As $\|\varrho\|_\gamma \leqslant \|\varrho-\varrho_n\|_\gamma + \|\varrho_n\|_\gamma$ for all n, also $\|\varrho\|_\gamma \leqslant 1$. □

All density operators ϱ satisfy

$$1 = \mathrm{Tr}(\varrho) = \|\varrho\|_1 \leqslant \|\varrho\|_\gamma$$

with equality if ϱ is separable. Thus one might tentatively consider the difference $\|\varrho\|_\gamma - \|\varrho\|_1$ as a measure of nonseparability.

Proposition 4. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be finite-dimensional Hilbert spaces and let ϱ be a density operator on $\mathcal{H}_1\otimes\mathcal{H}_2$ with $\|\varrho\|_\gamma \leqslant 1$, then ϱ is separable.*

Proof. Let ϱ be a density operator on $\mathcal{H}_1\otimes\mathcal{H}_2$ with $\|\varrho\|_\gamma \leqslant 1$. We divide the proof of separability into two steps. Firstly we show that for every $\delta > 0$ there exist families $\{x_i(\delta)\}$ and $\{y_i(\delta)\}$ of trace class operators on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively such that $\varrho = \sum_i x_i(\delta)\otimes y_i(\delta)$, where the sum converges with respect to the trace class norm, and such that

$$\sum_i \|x_i(\delta)\|_1 \|y_i(\delta)\|_1 \leqslant \|\varrho\|_1 + \delta = 1 + \delta.$$

As $\varrho \in \mathcal{B}(\mathcal{H}_1) \otimes_\gamma \mathcal{B}(\mathcal{H}_2)$, there exist elements $\varrho_n(\delta) \in \mathcal{B}(\mathcal{H}_1) \otimes_{\text{alg}} \mathcal{B}(\mathcal{H}_2)$, where $n \in \mathbb{N}$, such that

$$\|\varrho - \varrho_n(\delta)\|_\gamma < \frac{1}{2^{n+3}}\delta.$$

Consequently, $\|\varrho_{n+1}(\delta) - \varrho_n(\delta)\|_\gamma < \frac{1}{2^{n+2}}\delta$ for all n. Therefore, $\varrho_{n+1}(\delta) - \varrho_n(\delta)$ can be written in the form

$$\varrho_{n+1}(\delta) - \varrho_n(\delta) = \sum_{k_{n+1}=1}^{m_{n+1}} x_{k_{n+1}}^{(n+1)}(\delta) \otimes y_{k_{n+1}}^{(n+1)}(\delta)$$

with $x_{k_{n+1}}^{(n+1)}(\delta) \in \mathcal{B}(\mathcal{H}_1)$, $y_{k_{n+1}}^{(n+1)}(\delta) \in \mathcal{B}(\mathcal{H}_2)$ and

$$\sum_{k_{n+1}=1}^{m_{n+1}} \|x_{k_{n+1}}^{(n+1)}(\delta)\|_1 \|y_{k_{n+1}}^{(n+1)}(\delta)\|_1 \leqslant \frac{1}{2^{n+2}}\delta.$$

Since

$$\|\varrho_0(\delta)\|_\gamma \leqslant \|\varrho\|_\gamma + \|\varrho_0(\delta) - \varrho\|_\gamma < \|\varrho\|_\gamma + \tfrac{1}{2}\delta$$

$\varrho_0(\delta)$ can be represented as

$$\varrho_0(\delta) = \sum_{k_0=1}^{m_0} x_{k_0}^{(0)}(\delta) \otimes y_{k_0}^{(0)}(\delta)$$

with $x_{k_0}^{(0)}(\delta) \in \mathcal{B}(\mathcal{H}_1)$, $y_{k_0}^{(0)}(\delta) \in \mathcal{B}(\mathcal{H}_2)$ and

$$\sum_{k_0=1}^{m_0} \|x_{k_0}^{(0)}(\delta)\|_1 \|y_{k_0}^{(0)}(\delta)\|_1 \leqslant \|\varrho\|_\gamma + \tfrac{1}{2}\delta.$$

Consequently,

$$\varrho = \varrho_0(\delta) + \sum_{n\in\mathbb{N}} (\varrho_{n+1}(\delta) - \varrho_n(\delta)) \tag{3}$$

$$= \sum_{n\in\mathbb{N}} \sum_{k_n=1}^{m_n} x_{k_n}^{(n)}(\delta) \otimes y_{k_n}^{(n)}(\delta). \tag{4}$$

Thus we arrive at

$$1 = \|\varrho\|_1 \leqslant \sum_{n\in\mathbb{N}} \sum_{k_n=1}^{m_n} \|x_{k_n}^{(n)}(\delta)\|_1 \|y_{k_n}^{(n)}(\delta)\|_1 \leqslant \|\varrho\|_\gamma + \delta \leqslant 1 + \delta \tag{5}$$

which concludes the first part of our proof of proposition 4. By virtue of (5) the sequence $\{x_{k_n}^{(n)}(\frac{1}{N}) \otimes y_{k_n}^{(n)}(\frac{1}{N})\}_{N\in\mathbb{N}\setminus 0}$ is bounded with respect to the trace class norm for every n, k_n. Therefore, by possibly passing to a subsequence, we can assume that $\{x_{k_n}^{(n)}(\frac{1}{N}) \otimes y_{k_n}^{(n)}(\frac{1}{N})\}_N$ converges in trace class norm to a trace class operator $x_{k_n}^{(n)} \otimes y_{k_n}^{(n)}$ for $N \to \infty$. From (5) we infer that

$$\left\| \sum_n \sum_{k_n} x_{k_n}^{(n)} \otimes y_{k_n}^{(n)} \right\|_1 \leqslant \sum_n \sum_{k_n} \|x_{k_n}^{(n)}\|_1 \|y_{k_n}^{(n)}\|_1 = 1$$

and thus $\sum_n \sum_{k_n} x_{k_n}^{(n)} \otimes y_{k_n}^{(n)}$ is convergent.

If we let $\delta \in]0, 1]$, then $\|\varrho_{n+1}(\delta) - \varrho_n(\delta)\|_\gamma < \frac{1}{2^{n+2}}\delta \leqslant \frac{1}{2^{n+2}}$ and $\|\varrho_0(\delta)\|_\gamma < \|\varrho\|_\gamma + \frac{1}{2}\delta \leqslant \|\varrho\|_\gamma + \frac{1}{2}$. Thus we find that

$$\sup_\delta \|\varrho_0(\delta)\|_\gamma + \sum_n \sup_\delta \|\varrho_{n+1}(\delta) - \varrho_n(\delta)\|_\gamma < \infty.$$

Thus we conclude (Weierstraß convergence criterion) that the series (3) converges uniformly on]0, 1] and therefore we can interchange the infinite sums in (3) and (4) with the limit $N \to \infty$, arriving at

$$\begin{aligned} \varrho &= \lim_{N\to\infty} \sum_n \sum_{k_n} x_{k_n}^{(n)}(1/N) \otimes y_{k_n}^{(n)}(1/N) \\ &= \sum_n \sum_{k_n} \lim_{N\to\infty} (x_{k_n}^{(n)}(1/N) \otimes y_{k_n}^{(n)}(1/N)) \\ &= \sum_n \sum_{k_n} x_{k_n}^{(n)} \otimes y_{k_n}^{(n)}. \end{aligned}$$

Moreover, by (5),

$$\begin{aligned} 1 = |\operatorname{Tr}(\varrho)| &= \left| \sum_n \sum_{k_n} \operatorname{Tr}(x_{k_n}^{(n)}(\delta)) \operatorname{Tr}(y_{k_n}^{(n)}(\delta)) \right| \\ &\leqslant \sum_n \sum_{k_n} |\operatorname{Tr}(x_{k_n}^{(n)}(\delta)) \operatorname{Tr}(y_{k_n}^{(n)}(\delta))| \\ &\leqslant \sum_n \sum_{k_n} \|x_{k_n}^{(n)}(\delta)\|_1 \|y_{k_n}^{(n)}(\delta)\|_1 \\ &\leqslant \sum_n \sum_{k_n} \|x_{k_n}^{(n)}\|_1 \|y_{k_n}^{(n)}\|_1 + \delta \\ &= 1 + \delta \end{aligned}$$

we see that $|\operatorname{Tr}(x_{k_n}^{(n)}(\delta)) \times \operatorname{Tr}(y_{k_n}^{(n)}(\delta))|$ converges to $\|x_{k_n}^{(n)}\|_1 \|y_{k_n}^{(n)}\|_1$ for all k_n, n. Thus $\|x_{k_n}^{(n)}\|_1 \|y_{k_n}^{(n)}\|_1 = |\operatorname{Tr}(x_{k_n}^{(n)})| |\operatorname{Tr}(y_{k_n}^{(n)})|$ and therefore $\|x_{k_n}^{(n)}\|_1 = |\operatorname{Tr}(x_{k_n}^{(n)})|$ and $\|y_{k_n}^{(n)}\|_1 = |\operatorname{Tr}(y_{k_n}^{(n)})|$. This implies that we can choose all $x_{k_n}^{(n)}$ and $y_{k_n}^{(n)}$ as positive trace class operators. This proves that ϱ is separable. □

Putting all our results together we arrive at the main theorem of this paper.

Theorem 5. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be finite-dimensional Hilbert spaces and ϱ be a density operator on $\mathcal{H}_1 \otimes \mathcal{H}_2$. Then ϱ is separable if and only if $\|\varrho\|_\gamma = 1$.*

3. Conclusion

To conclude we have been able to prove a new mathematical separability criterion for density operators: a density operator ϱ on a finite-dimensional tensor product Hilbert space is separable if and only if $\|\varrho\|_\gamma = 1$. Our results also imply that the difference $\|\varrho\|_\gamma - \|\varrho\|_1 = \|\varrho\|_\gamma - 1$ may be considered as a quantitative measure of entanglement. In general it will be difficult to compute $\|\varrho\|_\gamma$ exactly, and accordingly theorem 5 is unlikely to provide a practical tool to decide whether a given density operator is separable or not without explicitly constructing a representation of the form (1). However, theorem 5 provides some principal insight into the structure of the space of density operators and therefore is of some interest in its own right. We have restricted ourselves to density operators on a tensor product Hilbert space of two finite-dimensional Hilbert spaces. It is straightforward, however, to generalize our results to the situation of density operators defined on a tensor product of more than two, but at most finitely many, finite-dimensional Hilbert spaces.

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